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ON THE DISTRIBUTION OF ADDITIVE ARITHMETIC FUNCTIONS ON THE POLYNOMIAL RING

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Abstract. We give a result on the density of certain additive arithmetic functions which count the number of restricted irreducible polynomials on the polynomial ring.

1. Introduction

The investigation of density of additive functions, the so-called local theorems in classical probabilistic number theory, was originated by Hardy and Ramanujan [1] and is continued by Erdős, Halász [2] and others. Especially, Halász applied the quantitative mean-value theorems in this investigation.

Halász [2] proved the theorem that is concerned with the "local" distribution of a certain completely additive function g defined in terms of a given set $\mathfrak{P}$ of primes as follows: $g(n)$ is the total number of prime divisors p of n such that $p \in \mathfrak{P}$, with multiplicity counted. Let $E(x) = \sum_{p\leq x, p\in\mathfrak{P}} p^{-1}$ and $N(m,x) = \sum_{n\leq x, g(n)=m} 1$. We assume that $\mathfrak{P}$ satisfies $E(x) \to \infty$ as $x \to \infty$.

Theorem 1.1.

$$N(m,x) = x\frac{E^m(x)}{m!}e^{-E(x)}\Big(1 + O(\frac{|m-E(x)|}{E(x)}) + O(\frac{1}{\sqrt{E(x)}})\Big)$$

uniformly in m and x for $\delta \leq m/E(x) \leq 2-\delta$, $E(x) \geq 2$, with any fixed $\delta > 0$.

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This conclusion refines earlier results of Kubilius [4]. Zhang [5] extended Theorem 1.1 to additive functions on additive arithmetic semigroups (see [6, Chapter 1]). From the above theorem, we easily have

$$N(m,x) \ll x\frac{E^m(x)}{m!}e^{-E(x)}, \text{ if } \delta \le \frac{m}{E(x)} \le 2-\delta$$

uniformly for the integer m. In [3], Halász extended the above upper bound to a larger range of $m/E(x)$:

THEOREM 1.2.

$$N(m,x) \ll x\frac{E^m(x)}{m!}e^{-E(x)}, \text{ if } 0 \le \frac{m}{E(x)} \le 2-\delta$$

In this paper, we give an analogous result for Theorem 1.2 in the polynomial ring case. Our proof is elementary and similar to that of Halász [3].

2. The distribution of additive arithmetic functions on the polynomial ring

An additive arithmetical semigroup $\mathcal{G}$ has a countable free generating set $\mathcal{P}$ of "primes" and a degree mapping $\partial\colon G \to \mathbb{N}\cup\{0\}$ satisfying $\partial(ab) = \partial(a)+\partial(b)$. The counting function $G(n) = \#\{a \in \mathcal{G},\, \partial(a) = n\}$ is assumed to satisfy

$$G(n) = Aq^n + O\left(\frac{q^n}{n^\gamma}\right),$$

where $A > 0$ and $\gamma > 2$.

For $\mathcal{P}^* \subset \mathcal{P}$, let

$$E(k) = \sum_{p\in\mathcal{P}^*,\, \partial(p)\le k} q^{-\partial(p)}$$

and

$$\Omega^*(a) = \#\{p \in \mathcal{P}^* : p \mid a\}$$

(counted with multiplicity). Let $N(m,k) = \sum_{\substack{\partial(a)=k,\\ \Omega^*(a)=m}} 1$ and δ be a real number satisfying $\max(\frac{1}{2}, 2-q) < \delta < 1$.

THEOREM 2.1 ([5], Theorem 5.1). *There is a positive constant c_9 such that*

$$\begin{aligned}\frac{1}{G(m)}N(m,k) &= \frac{E^m(k)}{m!}\exp\{-E(k)\}\cdot\\ &\Big\{1+\frac{\sqrt{(\Gamma(\delta-\frac{1}{2}))}}{(\delta-\frac{1}{2})^5}\cdot O\Big(\frac{|m-E(k)|}{E(k)}+\frac{1}{\sqrt{E(k)}}\Big)\\ &+O\Big(\exp\Big\{-c_9\big(\delta-\frac{1}{2}\big)^2\sum_{\substack{p\in\mathcal{P}^*,\partial(p)\le k,\\ \partial(p)\ \text{even}}} q^{-\partial(p)}\Big\}\Big)\Big\}\end{aligned}$$

for $E(k)>(q-2+\delta)^3$ and $\delta\le\frac{m}{E(k)}\le 2-\delta$. Here c_9 and the O-constants are independent of δ.

Let $\mathbb{F}_q(T)$ be the rational function field over a finite field $\mathbb{F}_q$. Throughout this paper, let $\mathcal{G}$ be the set of monic polynomials in $\mathbb{F}_q(T)$. Then $\mathcal{G}$ becomes an additive arithmetic semigroup with $\mathcal{P}$ being monic irreducible polynomials and $\partial(a)$ being the degree of a polynomial a. Clearly $G(k)=q^k$. Suppose that $\mathcal{P}^*$ satisfy $E(x)\to\infty$ as $x\to\infty$. From the theorem 2.1, we have

$$N(m,k)\ll q^k\frac{E^m(k)}{m!}e^{-E(k)}\ \text{if}\ \delta\le\frac{m}{E(k)}\le 2-\delta$$

uniformly for the integer k.

In this paper, we give an analogous result for Theorem 1.2 in the polynomial ring case:

THEOREM 2.2.

$$N(m,k)\ll q^k\frac{E^m(k)}{m!}e^{-E(k)}\ \textit{if}\ 0\le\frac{m}{E(k)}\le 2-\delta$$

uniformly for the integer k.

Proof. Let $f(a)=z^{\Omega^*(a)}$ for $a\in\mathcal{G}$. Then $f(a)$ is the completely multiplicative function on $\mathcal{G}$ defined by

$$f(p)=\begin{cases} z, & \text{if } p\in\mathcal{P}^*;\\ 1, & \text{otherwise.}\end{cases}$$

Following [3], we define

$$F(z,\sigma)=\sum_{a\in\mathcal{G}}\frac{f(a)}{|a|^\sigma}=\sum_{a\in\mathcal{G}}\frac{z^{\Omega^*(a)}}{|a|^\sigma}\quad (z=re^{i\theta}, r\le 2-\delta).$$

This is the Dirichlet series associated to f (for example [8, section 2]). Note that

$$\sum_{\substack{a\in\mathcal{G}\\ \Omega^*(a)=m}} \frac{1}{|a|^\sigma} = \sum_{k=1}^{\infty} \sum_{\substack{\partial(a)=k\\ \Omega^*(a)=m}} \frac{1}{|a|^\sigma} = \sum_{m=1}^{\infty} \frac{N(m,k)}{q^{k\sigma}} \tag{2.1}$$

and

$$\begin{aligned} \sum_{\substack{a\in\mathcal{G}\\ \Omega^*(a)=m}} \frac{1}{|a|^\sigma} &= \sum_{a\in\mathcal{G}} \frac{1}{|a|^\sigma} \cdot \frac{1}{2\pi i} \int_{|z|=r} z^{\Omega^*(a)-m-1} dz \\ &= \frac{1}{2\pi i} \int_{|z|=r} \frac{F(z,\sigma)}{z^{m+1}} dz. \end{aligned} \tag{2.2}$$

Since f is multiplicative, we have

$$\begin{aligned} F(z,\sigma) &= \prod_{p\in\mathcal{P}} \Big(\sum_{j=0}^{\infty} \frac{f(p^j)}{|p|^{j\sigma}}\Big) \\ &= \prod_{p\in\mathcal{P}^*} \Big(\sum_{j=0}^{\infty} \frac{z^j}{|p|^{j\sigma}}\Big) \prod_{p\notin\mathcal{P}^*} \Big(\sum_{j=0}^{\infty} \frac{1}{|p|^{j\sigma}}\Big) \\ &= \prod_{p\in\mathcal{P}^*} \Big(\frac{1}{1-\frac{z}{|p|^\sigma}}\Big) \prod_{p\notin\mathcal{P}^*} \Big(\frac{1}{1-\frac{1}{|p|^\sigma}}\Big) \\ &= \exp\Big(\sum_{p\in\mathcal{P}^*} \sum_{j=1}^{\infty} \frac{z^j}{jp^{j\sigma}} + \sum_{p\notin\mathcal{P}^*} \sum_{j=1}^{\infty} \frac{1}{jp^{j\sigma}}\Big). \end{aligned}$$

As in [3], estimating

$$\frac{F(z,\sigma)}{F(r,\sigma)} = \exp\Big(\sum_{p\in\mathcal{P}^*} \sum_{j=1}^{\infty} \frac{z^j - r^j}{jp^{j\sigma}}\Big)$$

we have

$$\begin{aligned} \frac{F(z,\sigma)}{F(r,\sigma)} &= \exp\Big(\sum_{\substack{p\in\mathcal{P}^*\\ |p|\le x'}} \frac{z-r}{|p|} + O(r|\theta|)\Big) \\ &= \exp\big((z-r)E(\log_q x')\big)(1+O(r|\theta|)) \\ &= e^{(z-r)E(x)} + O(e^{r(\cos\theta-1)E(x)} r|\theta|). \end{aligned}$$

Here k' is defined by $\sigma - 1 = \frac{1}{\log_q k'}$ and $k = \log_q k'$.

By the same argument as in the proof of [3, Theorem], we have for $(m+1)/E(k) \le 2-\delta$,

$$\sum_{\substack{a\in\mathcal{G}\\ \Omega^*(a)=m}} \frac{1}{|a|^\sigma} = F(r,\sigma)\frac{E^m(k)}{m!}e^{-rE(k)}\Big(1+O\Big(\frac{1}{\sqrt{E(k)}}\Big)\Big).$$

Estimating as above we have

$$F(r,\sigma) = e^{(r-1)E(k)+O(1)}k.$$

Combining this results with (2.1),

$$\sum_{u=1}^{\infty}\frac{N(m,u)}{q^{u\sigma}} \le c_1\frac{E^m(k)}{m!}e^{-E(k)}k$$

$$(m+1\le(2-\delta)E(x), \sigma-1=\frac{1}{x}).$$

If $u\le k$, then $q^{u(\sigma-1)} \le (q^k)^{1/k} = q$ and

$$\sum_{u\le k}\frac{N(m,u)}{q^u} \le q\sum_{u=1}^{\infty}\frac{N(m,u)}{q^{u\sigma}} \le c_2\frac{E^m(k)}{m!}e^{-E(k)}k. \tag{2.3}$$

Now introducing the Mangoldt function

$$\Lambda(a) = \begin{cases}\partial(P) & \text{if } a=P^r\\ 0 & \text{otherwise,}\end{cases}$$

as in the [3], we have

$$\sum_{\substack{\partial(a)=u\\ \Omega^*(a)=m}} \partial(a) \le \sum_P \partial(P)\Big(N(m-1,u-\partial(P))+N(m,u-\partial(P))\Big)+O(q^u).$$

Now

$$\begin{aligned}
kN(m,k) &= \sum_{\substack{\partial(a)=k\\ \Omega^*(a)=m}}\partial(a) \le \sum_{\substack{\partial(a)\le k+1\\ \Omega^*(a)=m}}\partial(a)\\
&\le \sum_{u\le k+1}\sum_P \partial(P)\Big(N(m-1,u-\partial(P))+N(m,u-\partial(P))\Big)+O(q^k)\\
&= \sum_P \partial(P)\sum_{u\le k+1-\partial(P)}\Big(N(m-1,u)+N(m,u)\Big)+O(q^k)\\
&= \sum_{u\le k}\Big(N(m-1,u)+N(m,u)\Big)\sum_{\partial(P)\le k+1-u}\partial(P)+O(q^k).
\end{aligned}$$

Since $\sum_{\partial(P)\le k+1-u}\partial(P)\le c_3q^k/q^u$ (see [8, Theorem 2.2]), we have from (2.3)

$$kN(m,k)\le c_4q^k\Big(\frac{E^{m-1}(k)}{(m-1)!}+\frac{E^m(k)}{m!}\Big)e^{-E(k)}k+O(q^k).$$

Therefore we have

$$N(m,k)\le c_5\Big(q^k\frac{E^m(k)}{m!}e^{-E(k)}+\frac{q^k}{k}\Big)$$

under the condition $(m+1)/E(k)\le 2-\delta$ which, δ being arbitary, is the same as that in our theorem. It remains to show that q^k/k is superflous, which is followed by the same argument as in the ring of integer case [3, Theorem]. It completes the proof. □

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